

# Index hypergeometric transform and imitation of analysis of Berezin kernels on hyperbolic spaces

NERETIN YU.A.<sup>1</sup>

The index hypergeometric transform (it is also called the Olevsky transform and the Jacobi transform) generalizes the spherical transforms in  $L^2$  on rank 1 symmetric spaces (i.e. real, complex and quaternionic Lobachevsky spaces). The purpose of the present work is to obtain properties of the index hypergeometric transform imitating the analysis of Berezin kernels on rank 1 symmetric spaces.

We also discuss a problem of explicit construction of a unitary operator identifying the space  $L^2$  and a Berezin space. The problem is reduced to some integral expression (the  $\Lambda$ -function) that apparently can't be expressed in a finite form in terms of standard special functions (only for some special values of parameters this expression can be reduced to the so-called Volterra type special functions). We investigate some properties of this expression. We show that for some series of symmetric spaces of large rank the operator of a unitary equivalence can be expressed through the determinant of some matrix consisting of the  $\Lambda$ -functions.

**0.1. Mehler–Weyl–Titchmarsh–Olevsky index hypergeometric transform.** Fix  $b, c > 0$ . Consider the integral transform  $J_{b,c}$  defined by the formula

$$\begin{aligned} g(s) &= J_{b,c}f(s) = \\ &= \frac{1}{\Gamma(b+c)} \int_0^\infty f(x) {}_2F_1(b+is, b-is; b+c; -x) x^{b+c-1} (1+x)^{b-c} dx, \end{aligned} \quad (0.1)$$

where  ${}_2F_1(\cdot)$  is the Gauss hypergeometric function. The inverse operator  $J_{b,c}^{-1}$  is given by

$$\begin{aligned} f(x) &= J_{b,c}^{-1}g(x) = \\ &= \frac{1}{\pi\Gamma(b+c)} \int_0^\infty g(s) {}_2F_1(b+is, b-is; b+c; -x) \left| \frac{\Gamma(b+is)\Gamma(c+is)}{\Gamma(2is)} \right|^2 ds. \end{aligned} \quad (0.2)$$

This pair of the inverse integral transforms was discovered by Weyl [38] in 1910.<sup>2</sup> Weyl's result didn't attract serious interest and again these transforms appeared in the works of Titchmarsh [35] in 1946 and M.N.Olevsky [30] 1949. Spherical transforms for all Riemannian noncompact symmetric spaces of rank 1 have the form (0.1) for some special values of the parameters  $b, c$ . This was the main reason of the interest in the transform (0.1) in subsequent years.

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<sup>2</sup>The partial case  $b = 1/2, c = 1$  (Mehler–Fock transform) was discovered by Mehler [21] in 1881.

For a detailed information on the transform (0.1) see the papers of Koornwinder and Flensted-Jensen [10], [16]–[18], see also the books of Yakubovich, Luchko [39], Samko, Kilbas, Marichev [33] and Tables of Prudnikov, Brychkov, Marichev [32], vol.5. Koornwinder's survey [17] contains a large bibliography.

For various generalizations of the transform (0.1) see, for instance, the works of Yakubovich, Luchko [39] (the Wimp transform with  $G$ -function), Koelink, Stockman [15] ( $q$ -analogs), Heckman, Opdam [12], [11], [31] (multivariate analogues).

In mathematical literature there exist the following terms for the transform (0.1)

- the index hypergeometric transform
- the generalized Fourier transform
- the Olevsky transform
- the Jacobi transform<sup>3</sup>
- the Fourier–Jacobi transform

**0.2. Contents.** The space  $L^2$  on a noncompact Riemannian symmetric space  $G/K$  admits a natural deformation discovered (in the case of Hermitian symmetric spaces) by Berezin [2], see also [27], [29], [5], [13] [24]–[26] and a bibliography in [26]. For small values of the deformation parameter (or for large values of our parameter  $\theta$ , see §3) the Berezin representations of the group  $G$  are equivalent to the natural representation of  $G$  in  $L^2(G/K)$ .

The Plancherel formula for the Berezin deformation for rank 1 spaces was obtained by van Dijk and Hille in [5] and in the general case by the author [24]–[26].

One of purposes of the present work is to apply these results to the theory of the index hypergeometric transform.

The second purpose is to obtain an explicit construction of a unitary intertwining operator from  $L^2(G/K)$  to a Berezin space for large values of the parameter  $\theta$ . We also apply the theory of the index hypergeometric transform for investigation of this operator.

Our §1 contains preliminaries on the index hypergeometric transform.

In §2 we calculate the images of the operators

$$f(x) \mapsto xf(x); \quad f(x) \mapsto x(x+1)\frac{d}{dx}f(x)$$

under the index hypergeometric transform. We show that these operators correspond to difference operators in the imaginary direction with respect to the variable  $s$ . For instance, the image of the multiplication by  $x$  is

$$Pg(s) = \frac{(b-is)(c-is)}{(-2is)(1-2is)}(g(s+i)-g(s)) + \frac{(b+is)(c+is)}{(2is)(1+2is)}(g(s-i)-g(s))$$

Although these results are very simple, I couldn't find the statement on  $x(x+1)\frac{d}{dx}$  in a literature. The statement on  $x$  is a very partial case of one recent Cherednik theorem [3].

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<sup>3</sup>The term was introduced by Koornwinder by an association with expansions in Jacobi polynomials.

In §3 we give a brief survey of analysis of Berezin kernels on rank 1 symmetric spaces. Below we imitate this section on the level of the index hypergeometric transform.

In §4 we give preliminaries on the continuous dual Hahn polynomials.

In §5 we obtain a family of nonstandard Plancherel formulas. Precisely, we show that the index transform is a unitary operator from some space of holomorphic functions in a disk (or in a half-plane) to the space  $L^2$  on a half-line with respect to some measure. This section is an imitation of the works [5], [24] on a level of special functions of one variable.

In §6 we define the function

$$\Lambda_{b,c}^a(x) = \frac{1}{\pi\Gamma(b+c)} \int_0^\infty \frac{\Gamma(a+is)\Gamma(b+is)\Gamma(b-is)\Gamma(c+is)\Gamma(c-is)}{\Gamma(2is)\Gamma(-2is)} \times \\ \times {}_2F_1(b+is, b-is; b+c; -x) ds$$

Our purpose is to construct a unitary intertwining operator from  $L^2$  to a Berezin space. Apparently, the function  $\Lambda$  can't be expressed explicitly by means of standard special functions (with exception of some special values of  $b, c$ , when  $\Lambda$  can be reduced to the rare Volterra type special functions). We also try to understand an answer to the following informal question: is it natural to consider the  $\Lambda$ -function as a "new" special function? We find a collection of identities with  $\Lambda$ . In §7 we construct some orthogonal bases in  $L^2$  on the half-line by means of the  $\Lambda$ -function.

In §8 we present (without a proof) an explicit construction of a unitary operator from  $L^2$  to the Berezin deformation of  $L^2$  for the symmetric spaces

$$U(p, q)/U(p) \times U(q).$$

It turned out that this integral operator is expressed by a determinant of a matrix consisting of the  $\Lambda$ -functions. This section explains the purposes of introduction of the  $\Lambda$ -function.

**0.3. Notation.**  $\mathbb{R}_+$  denotes the half-line  $x \geq 0$ .

$${}_2F_1(a, b; c; x) := \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n n!} x^n$$

is the Gauss hypergeometric function.

$${}_pF_q \left[ \begin{matrix} a_1, \dots, a_p \\ c_1, \dots, c_q \end{matrix}; x \right] := \sum_{n=0}^{\infty} \frac{(a_1)_n \dots (a_p)_n}{(c_1)_n \dots (c_q)_n n!} x^n$$

is the generalized hypergeometric function.

## §1. Preliminaries

**1.1. Definition.** Fix  $b, c > 0$ . Let  $f(x)$  be a function on the half-line  $\mathbb{R}_+$ . Its *index hypergeometric transform*  $J_{b,c}f$  (we also use the notation  $[\hat{f}]_{b,c}$ )

is given by

$$\begin{aligned} g(s) &= J_{b,c}f(s) = [\widehat{f}(s)]_{b,c} = \\ &= \frac{1}{\Gamma(b+c)} \int_0^\infty f(x) {}_2F_1(b+is, b-is; b+c; -x) x^{b+c-1} (1+x)^{b-c} dx. \end{aligned} \quad (1.1)$$

The inversion formula for the index hypergeometric transform is

$$\begin{aligned} f(x) &= J_{b,c}^{-1}g(x) = \\ &= \frac{1}{\pi \Gamma(b+c)} \int_0^\infty g(s) {}_2F_1(b+is, b-is; b+c; -x) \left| \frac{\Gamma(b+is)\Gamma(c+is)}{\Gamma(2is)} \right|^2 ds. \end{aligned} \quad (1.2)$$

This is equivalent to the following statement:

$J_{b,c}$  is a unitary operator

$$L^2(\mathbb{R}_+, x^{b+c-1} (1+x)^{b-c} dx) \rightarrow L^2(\mathbb{R}_+, \left| \frac{\Gamma(b+is)\Gamma(c+is)}{\Gamma(2is)} \right|^2 dx).$$

This property (*the Plancherel formula*) in the explicit form is

$$\int_0^\infty f_1(x) \overline{f_2(x)} x^{b+c-1} (1+x)^{b-c} dx = \frac{1}{\pi} \int_0^\infty [\widehat{f}_1(s)]_{b,c} \overline{[\widehat{f}_2(s)]_{b,c}} \left| \frac{\Gamma(b+is)\Gamma(c+is)}{\Gamma(2is)} \right|^2 ds.$$

Behavior of the operator  $J_{b,c}$  in different functional spaces are known in details, see the survey [17], and also [39].

REMARK. Weyl's ([38]) proof of the inversion formula (1.2) is relatively simple. Consider the hypergeometric differential operator

$$D := x(x+1) \frac{d^2}{dx^2} + [(c+b) + (2b+1)x] \frac{d}{dx} + b^2$$

on  $\mathbb{R}_+$ . The functions  ${}_2F_1(b+is, b-is; b+c; -x)$  are the eigenfunctions of  $D$ . There exists an explicit formula for the resolution  $R(\lambda) = (D - \lambda)^{-1}$  of the operator  $D$  (this is equivalent to an explicit expression for the Green function, see Myller-Lebedeff's works [22]–[23]). If we know the resolution, then we can easily find the spectral expansion by the formula

$$D = \frac{1}{2\pi i} \int \lim_{\varepsilon \rightarrow 0} [R(\lambda + i\varepsilon) - R(\lambda - i\varepsilon)] d\lambda$$

A detailed discussion of this reception is contained in Dunford, Schwartz [6]. For differential operators (and in particular for the hypergeometric operator) this method is present in Titchmarsh's book [35].

For other proofs of the inversion formula see [30], [16]–[18], [33], [39].

REMARK. Obviously,

$$[\widehat{Df}] = s^2 [\widehat{f}(s)]$$

## 1.2. Holomorphic extension to the strip.

LEMMA 1.1. Let  $f$  be integrable on  $\mathbb{R}_+$  and

$$f(x) = o(x^{-\alpha-\varepsilon}), \quad x \rightarrow +\infty$$

where  $\varepsilon > 0$ . Then  $[\widehat{f}(s)]_{b,c}$  is holomorphic in the strip

$$|\operatorname{Im} s| < \alpha - b$$

PROOF. This is a consequence of the following asymptotics for the hypergeometric function (see [9], vol. 1, (2.3.2.9)) as  $x \rightarrow +\infty$

$${}_2F_1(b+is, b-is; b+c; -x) = \lambda_1 x^{-b+is} + \lambda_2 x^{-b-is} + O(x^{-b+is-1}) + O(x^{-b-is-1}),$$

where  $2s \notin \mathbb{Z}$  and  $\lambda_1, \lambda_2$  are some constants (for  $2s \in \mathbb{Z}$  there appears the factor  $\ln x$  in the front of the leading term).

REMARK. Assume a continuous function  $f(x)$  satisfies the condition

$$f(x) = o(x^{-b-\varepsilon}), \quad x \rightarrow +\infty.$$

Then  $f(x)$  is an element of  $L^2(\mathbb{R}_+)$  with respect to the measure  $x^{b+c-1}(1+x)^{b-c}dx$ . We stress that this condition is also sufficient for existence of analytic continuation to a narrow strip.

**1.3. Fourier transform.** For some special values of the parameters  $b, c$  the function  ${}_2F_1(b+is, b-is; b+c; -x)$  is an elementary function. In particular (see [9], vol. 1, (2.8.11–12)),

$${}_2F_1(is, -is; \frac{1}{2}; -x) = \cos(2s \operatorname{arcsh} \sqrt{x}), \quad (1.3)$$

$${}_2F_1(\frac{1}{2} + is, \frac{1}{2} - is; \frac{3}{2}; -x) = \frac{\sin(2s \operatorname{arcsh} \sqrt{x})}{2s\sqrt{x}}. \quad (1.4)$$

Hence,

$$\begin{aligned} J_{0,1/2}f(s) &= \frac{1}{\Gamma(1/2)} \int_0^\infty f(x) \cos(2s \operatorname{arcsh} \sqrt{x}) x^{-1/2} (1+x)^{-1/2} dx, \\ J_{1/2,1}f(x) &= \frac{1}{\Gamma(3/2)} \int_0^\infty f(x) \frac{\sin(2s \operatorname{arcsh} \sqrt{x})}{2s\sqrt{x}} x^{1/2} (1+x)^{-1/2} dx. \end{aligned}$$

The substitution  $x = \operatorname{sh}^2 y$  reduces these integrals to the form

$$\begin{aligned} J_{0,1/2}f(s) &= \frac{2}{\Gamma(1/2)} \int_0^\infty f(\operatorname{sh}^2 y) \cos(2sy) dy, \\ J_{1/2,1}f(s) &= \frac{1}{\Gamma(3/2)} \int_0^\infty f(\operatorname{sh}^2 y) \frac{\operatorname{sh} y \sin(2sy)}{2s} dy, \end{aligned}$$

i.e. the operator  $J_{0,1/2}$  (resp.  $J_{1/2,1}$ ) differs from the cos-Fourier transform (resp. the sin-Fourier transform) by a nonessential variation of the notation.

Notice that the following functions also are elementary

$${}_2F_1(k+is, k-is; \frac{1}{2}+k+l; -x); \quad {}_2F_1(\frac{1}{2}+k+is, \frac{1}{2}+k-is; \frac{3}{2}+k+l; -x).$$

for integer values of  $k, l$ . They can be obtained from (1.3)–(1.4) by an application of suitable explicit differential operators (see [9], vol. 1, 2.8, formulas (20), (22), (24), (27)).

**1.4. Spherical transform.** Denote by  $\mathbb{K}$  the real numbers  $\mathbb{R}$ , the complex numbers  $\mathbb{C}$ , or the quaternions  $\mathbb{H}$ . Denote by  $r$  the dimension of the field  $\mathbb{K}$ . By  $\mathbb{K}^n$  we denote the  $n$ -dimensional space over  $\mathbb{K}$  with the standard scalar product

$$\langle z, u \rangle = \sum z_j \bar{u}_j.$$

Denote by  $U(1, n; \mathbb{K})$  the *pseudounitary group* over  $\mathbb{K}$ , i.e. the group of all  $(1+n) \times (1+n)$  matrices  $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$  over  $\mathbb{K}$ , satisfying the condition

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix}^* = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

The standard notations for  $U(1, n; \mathbb{K})$  in the cases  $\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}$  are respectively:  $O(1, n)$ ,  $U(1, n)$ ,  $Sp(1, n)$ .

By  $B_n(\mathbb{K})$  we denote the unit ball  $\langle z, z \rangle < 1$  in  $\mathbb{K}^n$ . Let  $S^{rn-1}$  be the unit sphere  $\langle z, z \rangle = 1$ . The group  $U(1, n; \mathbb{K})$  acts on  $B_n(\mathbb{K})$  by the linear fractional transformations

$$z \mapsto z^{[g]} := (a + zc)^{-1}(b + zd). \quad (1.5)$$

The stabilizer  $K$  of the point  $0 \in B_n(\mathbb{K})$  consists of the matrices

$$\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \quad |a| = 1, \quad d \in U(n; \mathbb{K}). \quad (1.6)$$

Hence  $B_n(\mathbb{K})$  is a homogeneous space,

$$B_n(\mathbb{K}) = U(1, n; \mathbb{K}) / U(1; \mathbb{K}) \times U(n; \mathbb{K}).$$

The Jacobian of transformation (1.5) is

$$J(g; z) = |a + zc|^{-r(1+n)}.$$

The following formula can easily be checked

$$1 - \langle z^{[g]}, u^{[g]} \rangle = (a + zc)^{-1}(1 - \langle z, u \rangle)(\overline{a + uc})^{-1}.$$

This implies that the  $U(1, n; \mathbb{K})$ -invariant measure on  $B_n(\mathbb{K})$  has the form

$$dm(z) = (1 - \langle z, z \rangle)^{-(n+1)r/2} dz,$$

where  $dz$  denotes the Lebesgue measure on  $B_n(\mathbb{K})$ .

The group  $\mathrm{U}(1, n; \mathbb{K})$  acts in  $L^2(\mathrm{B}_n(\mathbb{K}), dm(z))$  by the substitutions

$$f(z) \mapsto f((a + zc)^{-1}(b + zd)). \quad (1.7)$$

Consider the problem of the decomposition of  $L^2(\mathrm{B}_n(\mathbb{K}), dm(z))$  into a direct integral of irreducible representations of the group  $\mathrm{U}(1, n; \mathbb{K})$ .

Let  $s \in \mathbb{R}$ . The representation  $T_s$  of (spherical) *principal unitary series* of the group  $\mathrm{U}(1, n; \mathbb{K})$  acts in  $L^2(S^{rn-1})$  by the operators

$$T_s \begin{pmatrix} a & b \\ c & d \end{pmatrix} f(h) = f((a + hc)^{-1}(b + hd)) |a + hc|^{-(n+1)r/2+1+is}, \quad (1.8)$$

where  $h \in S^{rn-1}$ .

Consider the operator  $A$  from the space  $L^2(\mathrm{B}_n(\mathbb{K}), dm(z))$  to the space of functions on  $S^{rn-1} \times \mathbb{R}_+$  given by

$$Af(h, s) = \int_{\mathrm{B}_n(\mathbb{K})} f(z) \frac{|1 - \langle z, h \rangle|^{-(n+1)r/2+1+is}}{|1 - \langle z, z \rangle|^{(n+1)r/4+1/2+is/2}} dz. \quad (1.9)$$

If  $f$  is transformed by (1.7), then  $G(h, s) := Af(h, s)$  is transformed by

$$G(h, s) \mapsto G((a + hc)^{-1}(b + hd), s) |a + hc|^{-(n+1)r/2+1+is}. \quad (1.10)$$

Now we want to construct the measure  $d\nu$  (it is called the *Plancherel measure*) on  $S^{rn-1} \times \mathbb{R}_+$  such that the operator  $A$  is a unitary operator

$$L^2(\mathrm{B}_n(\mathbb{K}), dm(z)) \rightarrow L^2(S^{rn-1} \times \mathbb{R}_+, d\nu)$$

(this will solve the problem of decomposition into a direct integral). Obviously, the factor  $|a + hc|$  ((1.8), (1.9)) is 1 if the matrix  $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$  has the form (1.6). Thus the required measure  $d\nu$  is invariant with respect to the group  $\mathrm{U}(n; \mathbb{K})$ . Hence the measure  $d\nu$  has the form

$$\sigma(s) ds dh.$$

To find the function  $\sigma(s)$ , we consider all functions  $f$  on  $\mathrm{B}_n(\mathbb{K})$  depending only on the radius  $|z|$ . It is convenient to introduce the variable

$$x = \frac{|z|^2}{1 - |z|^2}$$

and to assume  $f = f(x)$ . Then the corresponding function  $G(h, s)$  depends only in the variable  $s$  and after a simple calculation we obtain

$$G(s) = \text{const} \cdot \int_0^\infty f(x) {}_2F_1(b + is, b - is; b + c; -x) x^{b+c-1} (1 + x)^{b-c} dx, \quad (1.11)$$

where

$$b = (n + 1)r/4 - 1/2; \quad c = (n - 1)r/4 + 1/2. \quad (1.12)$$

The integral transform (1.11) defined on the space of  $U(n, \mathbb{K})$ -invariant functions on  $B_n(\mathbb{K})$  is called the *spherical transform*. We observe that the spherical transforms are special cases of the index hypergeometric transform. The inversion formula (1.2) gives the following density for the Plancherel measure

$$\sigma(s) = \left| \frac{\Gamma(b+is)\Gamma(c+is)}{\Gamma(2is)} \right|^2 \quad (1.13)$$

## §2. Correspondence of some operators

### 2.1. Operator of multiplication by $x$ .

THEOREM 2.1. Assume that a function  $f$  defined on  $\mathbb{R}^+$  is continuous and satisfies the condition

$$f(x) = o(x^{-b-1-\varepsilon}); \quad x \rightarrow +\infty. \quad (2.1)$$

Then

$$\widehat{[xf(x)]_{b,c}} = P[\widehat{f(x)}]_{b,c}, \quad (2.2)$$

where the difference operator  $Pg$  is given by

$$Pg(s) = \frac{(b-is)(c-is)}{(-2is)(1-2is)}(g(s+i)-g(s)) + \frac{(b+is)(c+is)}{(2is)(1+2is)}(g(s-i)-g(s)).$$

PROOF. By condition (2.1), there exists a holomorphic continuation of the function  $\widehat{f}$  to the strip

$$|\operatorname{Im} s| < 1 + \varepsilon.$$

The identity (2.2) is equivalent to the following identity for hypergeometric functions

$$\begin{aligned} x {}_2F_1(b+is, b-is; b+c; -x) &= \\ &= \frac{(b-is)(c-is)}{(-2is)(1-2is)} {}_2F_1(b-1+is, b+1-is; b+c; -x) - \\ &- \left[ \frac{(b-is)(c-is)}{(-2is)(1-2is)} + \frac{(b+is)(c+is)}{(2is)(1+2is)} \right] {}_2F_1(b+is, b-is; b+c; -x) + \\ &+ \frac{(b+is)(c+is)}{(2is)(1+2is)} {}_2F_1(b+1+is, b-1-is; b+c; -x). \end{aligned}$$

The last identity is equivalent to the following relation between associated hy-

pergeometric functions<sup>4</sup>

$$\begin{aligned}
-y {}_2F_1(p, q; r; y) &= \\
&= \frac{q(r-p)}{(q-p)(1+q-p)} {}_2F_1(p-1, q+1; r; y) - \\
&- \left[ \frac{q(r-p)}{(q-p)(1+q-p)} + \frac{p(r-q)}{(p-q)(1+p-q)} \right] {}_2F_1(p, q; r; y) + \\
&+ \frac{p(r-q)}{(p-q)(1+p-q)} {}_2F_1(p+1, q-1; r; y). \quad (2.3)
\end{aligned}$$

The last identity is absent in standard tables but it can easily be checked by equating of the coefficients of the Taylor expansion in  $y$ . Indeed,

$$\begin{aligned}
{}_2F_1(p-1, q+1; r; y) - {}_2F_1(p, q; r; y) &= \sum \left( \frac{(p-1)_k (q+1)_k}{(r)_k k!} - \frac{(p)_k (q)_k}{(r)_k k!} \right) = \\
&= \sum \frac{(p)_k (q)_k}{(r)_k k!} \left( \frac{(p-1)(q+k)}{(p+k-1)q} - 1 \right) = \sum \frac{(p)_k (q)_k}{(r)_k k!} \cdot \frac{-k(1+q-p)}{q(p+k-1)}.
\end{aligned}$$

Converting  ${}_2F_1(p+1, q-1; r; y) - {}_2F_1(p, q; r; y)$ , we reduce the right part of (2.3) to the form

$$\sum \frac{(p)_k (q)_k}{(r)_k k!} \left( \frac{-k(r-p)}{(q-p)(p+k-1)} - \frac{-k(r-q)}{(p-q)(q+k-1)} \right).$$

The last expression equals the left part of (2.3)

**2.2. Differentiation.** THEOREM 2.2. *Let  $f, f'$  be continuous and satisfying the decreasing conditions (2.2). Then*

$$\widehat{x(x+1)\frac{d}{dx}f}_{b,c} = H[\widehat{f}]_{b,c}, \quad (2.4)$$

where the difference operator  $H$  is given by

$$\begin{aligned}
Hg(s) &= \frac{(b-is)(b+1-is)(c-is)}{(-2is)(1-2is)} (g(s+i) - g(s)) + \\
&+ \frac{(b+is)(b+1+is)(c+is)}{(+2is)(1+2is)} (g(s-i) - g(s)) - (b+c)g(s).
\end{aligned}$$

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<sup>4</sup>For any triple of hypergeometric functions  ${}_2F_1(p, q; r; y)$ ,  ${}_2F_1(p+k_1, q+l_1; r+m_1; y)$ ,  ${}_2F_1(p+k_2, q+l_2; r+m_2; y)$  with integer  $k_j, l_j, m_j$  there exists a linear relation with coefficient, which are polynomial in  $ys$ .

PROOF.

$$\begin{aligned}
& \int_0^\infty \left\{ x(x+1) \frac{d}{dx} f(x) \right\} {}_2F_1(b+is, b-is; b+c; -x) x^{b+c-1} (1+x)^{b-c} dx = \\
&= - \int_0^\infty f(x) \frac{d}{dx} \left\{ x^{b+c} (1+x)^{b-c+1} {}_2F_1(b+is, b-is; b+c; -x) \right\} dx = \\
&= - \int_0^\infty f(x) \left\{ x(x+1) \frac{d}{dx} {}_2F_1(b+is, b-is; b+c; -x) + \right. \\
&\quad \left. + ((2b+1)x + (b+c)) {}_2F_1(b+is, b-is; b+c; -x) \right\} x^{b+c-1} (1+x)^{b-c} dx.
\end{aligned}$$

Now the statement is reduced to the identity

$$\begin{aligned}
& - \left\{ x(x+1) \frac{d}{dx} + (2b+1)x + (b+c) \right\} {}_2F_1(b+is, b-is; b+c; -x) = \\
&= \frac{(b-is)(b+1-is)(c-is)}{(-2is)(1-2is)} {}_2F_1(b-1+is, b+1-is; b+c; -x) - \\
&\quad - \left( \frac{(b-is)(b+1-is)(c-is)}{(-2is)(1-2is)} + \frac{(b+is)(b+1+is)(c+is)}{(+2is)(1+2is)} - b-c \right) \times \\
&\quad \times {}_2F_1(b+is, b-is; b+c; -x) + \\
&\quad + \frac{(b+is)(b+1+is)(c+is)}{(+2is)(1+2is)} {}_2F_1(b+1+is, b-1-is; b+c; -x).
\end{aligned}$$

This is an relation between associated hypergeometric functions

$$\begin{aligned}
& - \left\{ x(x+1) \frac{d}{dx} + (p+q+1)x + r \right\} {}_2F_1(p, q; r; -x) = \\
&= (x^2 + x) \frac{pq}{r} {}_2F_1(p+1, q+1; r+1; -x) - ((p+q+1)x + r) {}_2F_1(p, q; r; -x) = \\
&= \frac{q(q+1)(r-p)}{(1+q-p)(p-q)} ( {}_2F_1(p-1, q+1; r; -x) - {}_2F_1(p, q; r; -x) ) + \\
&\quad + \frac{p(p+1)(r-q)}{(1+p-q)(q-p)} ( {}_2F_1(p+1, q-1; r; -x) - {}_2F_1(p, q; r; -x) )
\end{aligned}$$

and can easily be checked by equating of the Taylor coefficients at  $x^k$ .

### §3. Berezin kernels

The space  $L^2$  on the ball  $B_n(\mathbb{K})$  (the ball  $B_n(\mathbb{K})$  is the symmetric space  $U(1, n; \mathbb{K})/U(1, \mathbb{K}) \times U(n, \mathbb{K})$ ) admits a natural deformation that will be described in this section (for more details see [5] for rank 1 symmetric spaces and [26] for symmetric spaces of arbitrary rank). The main purpose of the present work is an imitation of this deformation on a level of special functions.

**3.1. Positive definite kernels.** Recall the definition of a *positive definite kernel* (for detailed discussion and references see [26]). Let  $X$  be a set. A

function  $L(x, y)$  on  $X \times X$  is called *a positive definite kernel* if  $L(y, x) = \overline{L(x, y)}$  and for all  $x_1, \dots, x_k \in X$  the matrix

$$\det \begin{pmatrix} L(x_1, x_1) & \dots & L(x_1, x_k) \\ \vdots & \ddots & \vdots \\ L(x_k, x_1) & \dots & L(x_k, x_k) \end{pmatrix}$$

is positive semidefinite.

Let  $L$  be a positive definite kernel on a set  $X$ . Then there exists a Hilbert space  $H = H[L]$  and a system of vectors  $v_x \in H$ , where  $x$  ranges  $X$ , such that

1. the linear span of the vectors  $v_x$  is dense in  $H[L]$
2.  $\langle v_x, v_y \rangle = L(y, x)$ .

For each vector  $h \in H[L]$  we assign the function  $f_h(x)$  on  $X$  by

$$f_h(x) = \langle h, v_x \rangle$$

Thus the space  $H$  is realized as some space of functions on  $X$ . We denote this function space by  $H^\circ[L]$ . Obviously, the vectors  $v_a \in H[L]$  correspond to the functions

$$\varphi_a(x) = L(x, a)$$

and for each  $f \in H^\circ[L]$  we have the following equality (the *reproducing property*)

$$\langle f, \varphi_a \rangle_{H^\circ[L]} = f(a). \quad (3.1)$$

This identity gives some (constructive, but not completely) description of the scalar product in  $H^\circ[L]$ . It is said also that  $L$  is a *reproducing kernel* of the space  $H^\circ[L]$ .

**3.2. Berezin kernels.** The *Berezin kernel*  $L_\theta(z, u)$  on  $B_n(\mathbb{K})$  is defined by

$$L_\theta(z, u) = |1 - \langle z, u \rangle|^{-\theta}. \quad (3.2)$$

In the cases  $\mathbb{K} = \mathbb{R}, \mathbb{C}$  the kernel  $L_\theta(z, u)$  is positive definite for all  $\theta \geq 0$ , and in the case  $\mathbb{K} = \mathbb{H}$  the condition of positive definiteness is  $\theta \geq 2$ . The group  $U(1, n; \mathbb{K})$  acts in  $H^\circ(L_\theta)$  by the unitary operators

$$T_\theta \begin{pmatrix} a & b \\ c & d \end{pmatrix} f(z) = f((a + zc)^{-1}(b + zd))|a + z|^{-\theta}.$$

**3.3. Another version of the definition.** Consider the kernel on  $B_n(\mathbb{K})$  given by

$$M_\theta(z, u) = \frac{(1 - \langle z, z \rangle)^{\theta/2}(1 - \langle u, u \rangle)^{\theta/2}}{|1 - \langle z, u \rangle|^\theta}.$$

Consider the spaces  $H(M_\theta)$ ,  $H^\circ(M_\theta)$  associated with this kernel. The group  $U(1, n; \mathbb{K})$  acts in the space  $H^\circ[M_\theta]$  by substitutions by the formula

$$R_\theta \begin{pmatrix} a & b \\ c & d \end{pmatrix} f(z) = f((a + zc)^{-1}(b + zd)). \quad (3.3)$$

It is easily shown that the constructions of 3.2 and 3.3 are equivalent. The canonical unitary  $U(1, n; \mathbb{K})$ -intertwining operator  $H^\circ[L_\theta] \rightarrow H^\circ[M_\theta]$  is the operator of the multiplication by the function  $(1 - \langle z, z \rangle)^{\theta/2}$ .

**3.4. More material description of the space  $H^\circ(L_\theta)$ .** The cases  $\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}$  are similar and we will consider the case  $\mathbb{K} = \mathbb{R}$  as a basic example. Let us give a more constructive description of the space  $H^\circ(L_\theta)$  in this case (with an additional restriction  $\theta > n$ ).

For this aim we consider the space  $V_\theta$  of holomorphic functions in the ball  $B_n(\mathbb{C})$ , we equip this space with the scalar product

$$\langle f, g \rangle_{V_\theta} = \frac{\Gamma(\theta)}{\pi^n \Gamma(\theta - n)} \int_{B_n(\mathbb{C})} f(z) \overline{g(z)} (1 - \langle z, z \rangle)^{\theta - n - 1} dz.$$

The universal covering group of the group  $U(1, n) = U(1, n; \mathbb{C})$  acts in the space  $V_\theta$  by the unitary operators

$$\mathcal{U}_\theta \begin{pmatrix} a & b \\ c & d \end{pmatrix} f(z) = f((a + zc)^{-1}(b + zd))(a + zc)^{-\theta}.$$

REMARK. For an integer  $\theta$  the family of the operators  $\mathcal{U}_\theta$  defines a linear representation of the group  $U(1, n)$ . If  $\theta$  is not integer, then

$$(a + zc)^{-\theta} = (1 + zca^{-1})^{-\theta} e^{-\theta(\ln a + 2\pi k i)}.$$

It can easily be shown, that  $\langle ca^{-1}, ca^{-1} \rangle < 1$ , hence  $(1 + zca^{-1})^{-\theta}$  is canonically defined for  $\langle z, z \rangle < 1$ . Thus the operator  $\mathcal{U}_\theta \begin{pmatrix} a & b \\ c & d \end{pmatrix}$  is defined up to a factor  $e^{2\pi k i \theta}$ . Hence for a noninteger  $\theta$  we obtain a projective representation of the group  $U(1, n)$  or (this is one and the same) a linear representation of its universal covering group.

A simple calculation (with the Dirichlet integral, see, for instance, [1], 1.8) shows that the system of functions  $z_1^{k_1} \dots z_n^{k_n}$  forms an orthogonal basis in  $V_\theta$  and

$$\|z_1^{k_1} \dots z_n^{k_n}\|^2 = \frac{k_1! \dots k_n!}{(\theta)_{k_1 + \dots + k_n}}. \quad (3.4)$$

Consider the function

$$\varphi_a(z) := (1 - \sum z_j \bar{a}_j)^{-\theta} = \sum_{k_1, \dots, k_n} \frac{(\theta)_{k_1 + \dots + k_n} \bar{a}_1^{k_1} \dots \bar{a}_n^{k_n}}{k_1! \dots k_n!} z_1^{k_1} \dots z_n^{k_n}. \quad (3.5)$$

By equations (3.4)–(3.5), for any function  $f \in V_\theta$

$$\langle f, \varphi_a \rangle_{V_\theta} = f(a).$$

Thus the space  $V_\theta$  is defined by the positive definite kernel

$$K(z, u) = (1 - \langle z, u \rangle)^{-\theta}.$$

A function holomorphic in the ball  $B_n(\mathbb{C})$  is uniquely determined by its restriction to  $B_n(\mathbb{R})$ , hence we can consider the space  $V_\theta$  as a space of functions on  $B_n(\mathbb{R})$ . This is strictly the space defined by the Berezin kernel  $L_\theta$  on  $B_n(\mathbb{R})$ .

Emphasize that in the space  $H^\circ[L_\theta]$  on  $B_n(\mathbb{R})$  we have an action of the group  $U(1, n)$ , which is larger than  $O(1, n)$ <sup>5</sup>; for more details see [26].

**3.5. Plancherel formula.** Emphasize, that the group  $U(1, n; \mathbb{K})$  acts in the space  $L^2(B_n(\mathbb{K}))$  and in all spaces  $H^\circ(M_\theta)$  by the same formula (1.7), (3.3). Consider the following operator, which coincides with operator (1.9) up to a functional factor

$$R_\theta f(h, s) = \frac{1}{\Gamma(b+c)|\Gamma(\theta-b+is)|^2} \int_{B_n(\mathbb{K})} f(z) \frac{|1-\langle z, h \rangle|^{-(n+1)r/2+1+is}}{|1-\langle z, z \rangle|^{(n+1)r/4+1/2+is/2}} dz. \quad (3.6)$$

As was shown by van Dijk and Hille [5], for  $\theta > b$  the operator  $R_\theta$  is a unitary operator

$$H^\circ(M_\theta) \rightarrow L^2(S^{rn-1} \times \mathbb{R}_+, \tau(s) dh ds),$$

where  $dh$  is a Lebesgue measure on the sphere and

$$\tau(s) = \frac{1}{\Gamma(\theta)\Gamma(\theta-b+c)} \left| \frac{\Gamma(\theta-b+is)\Gamma(b+is)\Gamma(c+is)}{\Gamma(2is)} \right|^2. \quad (3.7)$$

REMARK. If we remove the expression  $|\Gamma^2(\theta-b+is)|^2$  from the denominator in (3.6), then in (3.7) the same expression will relocate from the numerator to the denominator. By some reasons, the normalization (3.6) is more convenient.

Below in §5 we obtain Theorem 5.3 imitating the Plancherel formula (3.7) on the level of the index hypergeometric transform, the van Dijk–Hille theorem is a special case of Theorem 5.3.

**3.6. Radial parts of Berezin kernels.** Again let us for definiteness consider the case  $\mathbb{K} = \mathbb{R}$ . Consider the space  $V_\theta$  of holomorphic functions defined in 3.4, and its subspace  $V_\theta^{O(n)}$  (*the radial Berezin space*) consisting of  $O(n)$ -invariant functions.

Obviously, the elements of the space  $V_\theta^{O(n)}$  have the form

$$g(z_1^2 + \cdots + z_n^2) = \sum c_p (z_1^2 + \cdots + z_n^2)^p.$$

Hence we can consider  $V_\theta^{O(n)}$  as a space of functions depending on one variable

$$u = z_1^2 + \cdots + z_n^2$$

lying in the disk  $|u| < 1$ .

---

<sup>5</sup> For the complex ball  $B_n(\mathbb{C})$ , the analogous overgroup is  $U(1, n) \times U(1, n)$ . For the quaternionic ball the overgroup is  $U(2, 2n)$

LEMMA 3.1. Vectors  $u^p$  form an orthogonal basis in the space  $V_\theta^{O(n)}$  and

$$\|u^p\|^2 = \frac{p! (n/2)_p}{(\theta/2)_p (\theta/2 + 1/2)_p}. \quad (3.8)$$

PROOF. Obviously, the vectors  $u^p$  are pairwise orthogonal, hence it is sufficient to calculate  $\|u^p\|^2$ . By orthogonality (3.4), we obtain

$$\begin{aligned} \|u^p\|^2 &= \|(z_1^2 + \dots + z_n^2)^p\|^2 = \sum_{k_1 \geq 0, \dots, k_n \geq 0, \sum k_j = p} \left( \frac{p!}{k_1! \dots k_n!} \right)^2 \|z_1^{2k_1} \dots z_n^{2k_n}\|^2 = \\ &= \sum_{k_1 \geq 0, \dots, k_n \geq 0, \sum k_j = p} \left( \frac{p!}{k_1! \dots k_n!} \right)^2 \frac{2k_1! \dots 2k_n!}{(\theta)_{2p}}. \end{aligned}$$

It is sufficient to evaluate

$$\sum_{k_1 \geq 0, \dots, k_n \geq 0, \sum k_j = p} \frac{2k_1! \dots 2k_n!}{(k_1! \dots k_n!)^2}. \quad (3.9)$$

Denote a summand of this sum by  $A_{k_1, \dots, k_n}$ . Consider the generating function

$$\begin{aligned} h(y_1, \dots, y_n) &:= \sum_{k_1 \geq 0, \dots, k_n \geq 0} A_{k_1, \dots, k_n} y_1^{k_1} \dots y_n^{k_n} = \\ &= \prod_{m=1}^n \sum_{k=0}^{\infty} \frac{2k!}{k! k!} y_m^k = \prod_{m=1}^n (1 - 4y_m)^{-1/2}. \end{aligned}$$

Expression (3.9) is the coefficient at  $y^p$  in the Taylor expansion of

$$h(y, \dots, y) = (1 - 4y)^{-n/2}$$

and this gives the required statement.

Spaces that imitate  $V_\theta^{O(n)}$  are defined below in 5.1.

**3.7. Ørsted problem.** As before, let  $b, c$  be defined by (1.12). For  $\theta > b$  the representation  $U(1, n; \mathbb{K})$  in  $H^\circ[L_\theta]$  is equivalent to the representation in  $L^2(B_n(\mathbb{K}))$ . There arises the following question:

*Is it possible to write explicitly a unitary intertwining operator*

$$S : L^2(B_n(\mathbb{K})) \rightarrow H^\circ[L_\theta] \quad ?$$

It is easy to obtain a general form of this operator. The operator  $S$  is a product of three operators

$$S = \text{const} \cdot R_\theta^{-1} M A,$$

where  $A$  is given by (1.9),  $R_\theta$  is defined by (3.6), and

$$M : L^2 \left( \mathbb{R}_+, \left| \frac{\Gamma(b+is)\Gamma(c+is)}{\Gamma(2is)} \right|^2 \right) \rightarrow L^2 \left( \mathbb{R}_+, \left| \frac{\Gamma(\theta-b+is)\Gamma(b+is)\Gamma(c+is)}{\Gamma(2is)} \right|^2 \right)$$

is the operator of division by a function  $\psi(s)$  satisfying the condition

$$|\psi(s)| = |\Gamma(a-b+is)|.$$

LEMMA 3.2. *The operator  $S$  is given by*

$$Sf(z) = \int_{B_n(\mathbb{K})} \Lambda \left( \frac{|z-u|^2}{(1-|z|^2)(1-|u|^2)} \right) f(u) dm(u),$$

where  $dm$  is the invariant measure on the ball and

$$\Lambda(x) = \text{const} \cdot \int_0^\infty \overline{\psi(s)} \left| \frac{\Gamma(b+is)\Gamma(c+is)}{\Gamma(2is)} \right|^2 {}_2F_1(b+is, b-is; b+c; -x) ds. \quad (3.10)$$

PROOF. Consider the function 1 from  $V_\theta$ . Its preimage with respect to the operator  $S$  can be easily evaluated, it is given by (3.10), where  $x = |z|^2/(1-|z|^2)$ . By the reproducing property, for any  $g \in V_\theta$  we have  $\langle g, 1 \rangle = g(0)$ . Since the operator  $S$  is unitary, any function  $f \in L^2$  satisfies

$$Sf(0) = \int_{B_n(\mathbb{K})} \Lambda \left( \frac{|z|^2}{1-|z|^2} \right) dm(z).$$

Now the kernel of the operator  $S$  can be reconstructed by invariance arguments.

Where arises a question, is it possible to find  $\psi(x)$  such that the integral (3.10) can be explicitly evaluated. For several years I tried to solve this problem and now I think that this is impossible. It seems that the simplest variant

$$\psi(s) = \Gamma(\theta-b-is)$$

is best.

In §6 I am trying to understand, is it natural to consider the integral (3.10) obtained in this way as a 'new' special function.

I am inclined to think that the answer is affirmative by the following 'metaphysical' reason. To be definite, assume  $\mathbb{K} = \mathbb{R}$ . Then the natural symmetry group  $U(1, n)$  of the space  $V_\theta$  is larger than the symmetry group  $O(1, n)$  of the space  $L^2(B_n(\mathbb{R}))$ . If we identify  $L^2$  with  $V_\theta$ , then we force the group  $U(1, n)$  to act in  $L^2(B_n(\mathbb{R}))$ . It is natural to think that enlarging of a symmetry group must imply a nontrivial analysis on the level of special functions.

There is also another fact that seems pleasant for me. For 3 series of semisimple groups  $O(n, \mathbb{C})$ ,  $Sp(n, \mathbb{C})$ ,  $U(p, q)$  the corresponding  $\Lambda$ -function can be expressed as a determinant consisting of  $\Lambda$ -functions (3.10) related to rank 1 symmetric spaces.

## §4. Hahn polynomials: preliminaries.

**4.1. Definition.**  $a, b, c > 0$ . The *continuous dual Hahn polynomials* (see [1], [36]) are given by

$$S_n(s^2; a, b, c) = (a+b)_n (a+c)_n {}_3F_2 \left[ \begin{matrix} -n, a+is, a-is \\ a+b, a+c \end{matrix}; 1 \right].$$

Obviously, the expression  $S_n(s^2; a, b, c)$  doesn't change if we transpose the parameters  $b, c$ . The Kummer formula (see [1], Corollary 3.3.5.)

$${}_3F_2 \left[ \begin{matrix} \alpha, \beta, \gamma \\ \delta, \varepsilon \end{matrix}; 1 \right] = \frac{\Gamma(\varepsilon)\Gamma(\delta + \varepsilon - \alpha - \beta - \gamma)}{\Gamma(\varepsilon - \alpha)\Gamma(\delta + \varepsilon - \beta - \gamma)} {}_3F_2 \left[ \begin{matrix} \alpha, \delta - \beta, \delta - \gamma \\ \delta, \delta + \varepsilon - \beta - \gamma \end{matrix}; 1 \right]$$

implies that  $S_n(s^2; a, b, c)$  is invariant with respect to arbitrary permutations of  $a, b, c$ .

**4.2. Orthogonality relations.** The Hahn polynomials form an orthogonal basis in  $L^2$  on half-line with respect to the weight

$$\left| \frac{\Gamma(a+is)\Gamma(b+is)\Gamma(c+is)}{\Gamma(2is)} \right|^2$$

and

$$\begin{aligned} \frac{1}{\pi} \int_0^\infty \left| \frac{\Gamma(a+is)\Gamma(b+is)\Gamma(c+is)}{\Gamma(2is)} \right|^2 S_n(s^2; a, b, c) S_m(s^2; a, b, c) dx = \\ = \Gamma(a+b+n)\Gamma(a+c+n)\Gamma(b+c+n) n! \delta_{m,n}. \end{aligned}$$

**4.3. Difference equations.** Consider the difference operator

$$\mathcal{L}y(s) = B(s)y(s+i) - (B(s) + D(s))y(s) + D(s)y(s-i), \quad (4.1)$$

where

$$\begin{aligned} B(s) &= \frac{(a-is)(b-is)(c-is)}{(-2is)(1-2is)}, \\ D(s) &= \frac{(a+is)(b+is)(c+is)}{(+2is)(1+2is)}. \end{aligned}$$

The Hahn polynomials are the eigenfunctions of this operator

$$\mathcal{L}S_n(s^2; a, b, c) = n \cdot S_n(s^2; a, b, c). \quad (4.2)$$

### 4.4. Index hypergeometric transform and Hahn polynomials.

LEMMA 4.1. *The image of the function  $(1+x)^{-a-b}$  under the transform  $J_{b,c}$  is*

$$\frac{\Gamma(a-is)\Gamma(a+is)}{\Gamma(a+b)\Gamma(a+c)}$$

PROOF. The statement is reduced to the table integral [7], 7.51.10.

LEMMA 4.2. *The image of the function*

$$\left(\frac{x}{x+1}\right)^n (1+x)^{-a-b}$$

*with respect to the transform  $J_{b,c}$  is*

$$\begin{aligned} \frac{\Gamma(a-is)\Gamma(a+is)}{\Gamma(a+b)\Gamma(a+c)} {}_3F_2 \left[ \begin{matrix} -n, a+is, a-is \\ a+b, a+c \end{matrix}; 1 \right] &= \\ &= \frac{|\Gamma(a+is)|^2}{\Gamma(a+b+n)\Gamma(a+c+n)} S_n(s^2; a, b, c). \end{aligned}$$

PROOF. We must evaluate the image of the function

$$\begin{aligned} \left(\frac{x}{x+1}\right)^n (1+x)^{-a-b} &= \left(\frac{1+x-1}{x+1}\right)^n (1+x)^{-a-b} = \\ &= \sum_{k=0}^n C_n^k (-1)^{n-k} (1+x)^{-a-c-n+k}. \end{aligned}$$

But the images of the functions  $(1+x)^{-a-c-n+k}$  were evaluated above.

## §5. Nonstandard Plancherel formulas

5.1. **Spaces  $H_{b,c}^a$ .** Fix positive numbers  $a, b, c$  satisfying the conditions

$$a > b, \quad a > c, \quad 2a > 1. \quad (5.1)$$

Consider the space  $W^a$  of holomorphic functions in the disk  $|z| < 1$  satisfying the condition

$$\iint_{|z|<1} |f(z)|^2 (1-|z|^2)^{2a-2} dz < \infty,$$

where  $dz$  denotes the Lebesgue measure in the disk.

Let us define the scalar product in  $W^a$  by the formula

$$\langle f, g \rangle = \frac{1}{\pi \Gamma(2a-1)} \iint_{|z|<1} f(z) \overline{g(z)} (1-|z|^2)^{2a-2} {}_2F_1(a-b, a-c; 2a-1; 1-|z|^2) dz.$$

We denote by  $W_{b,c}^a$  the Hilbert space obtained in this way.

REMARK. The spaces  $W_{b,c}^a$  as linear spaces don't depend on  $b, c$  and coincide with  $W^a$ . But the scalar products in these spaces are different.

LEMMA 5.1. *The functions  $z^k$  form an orthogonal basis in  $W_{b,c}^a$  and*

$$\langle z^k, z^k \rangle_{W_{b,c}^a} = \frac{k! \Gamma(b+c+k)}{\Gamma(a+b+k) \Gamma(a+c+k)}. \quad (5.2)$$

PROOF. Orthogonality of the functions  $z^k$  is obvious. Let us evaluate

$$\begin{aligned}
\langle z^k, z^k \rangle &= \\
&= \frac{1}{\pi \Gamma(2a-1)} \iint_{|z|<1} |z|^{2k} (1-|z|^2)^{2a-2} {}_2F_1(a-b, a-c; 2a-1; 1-|z|^2) dz = \\
&= \frac{2}{\Gamma(2a-1)} \int_0^1 r^{2k+1} (1-r^2)^{2a-2} {}_2F_1(a-b, a-c; 2a-1; 1-r^2) dr = \\
&= \frac{1}{\Gamma(2a-1)} \int_0^1 y^k (1-y)^{2a-2} {}_2F_1(a-b, a-c; 2a-1; 1-y) dy = \\
&= \frac{1}{\Gamma(2a-1)} \int_0^1 (1-v)^k v^{2a-2} {}_2F_1(a-b, a-c; 2a-1; v) dv
\end{aligned}$$

Then all is reduced to the table integral [7], 7.512.4.

REMARK. We observe that the radial Berezin spaces  $V_\theta^{\mathcal{O}(n)}$  defined in 3.6 are special cases of the spaces  $W_{b,c}^a$ :

$$b = n/4 - 1/4; \quad c = n/4 + 1/4; \quad a = \theta - n/4 + 1/4.$$

### 5.2. Reproducing kernels of the spaces $W_{b,c}^a$ .

LEMMA 5.2. *The reproducing kernel of the space  $W_{b,c}^a$  is*

$$K_{b,c}^a(z, u) = \frac{\Gamma(a+b)\Gamma(a+c)}{\Gamma(b+c)} {}_2F_1 \left[ \begin{matrix} a+b, a+c \\ b+c \end{matrix}; z\bar{u} \right].$$

PROOF. Obvious. Indeed, assume  $\varphi_w(z) = K_{b,c}^a(z, w)$ . Then for any holomorphic function  $f(z) = \sum_{k=0}^{\infty} c_k z^k$  we have

$$\begin{aligned}
\langle f, \varphi_w \rangle_{W_{b,c}^a} &= \sum_{k=0}^{\infty} c_k w^k \frac{\Gamma(a+b+k)\Gamma(a+c+k)}{k! \Gamma(b+c+k)} \langle z^k, z^k \rangle = \\
&= \sum_{k=0}^{\infty} c_k w^k = f(w).
\end{aligned}$$

REMARK. Kernels  $K_{b,c}^a(z, u)$  are special cases of the kernels introduced by Gross, Richards [8] and also special cases of the kernels considered by Odzijewicz [28].

**5.3. Plancherel formula.** Let  $g \in W_{b,c}^a$ . Consider the transform

$$\begin{aligned}
J_{b,c}^a g(s) &= \frac{1}{|\Gamma(a+is)|^2 \Gamma(b+c)} \int_0^\infty (1+x)^{-a-b} g\left(\frac{x}{x+1}\right) \times \\
&\quad \times {}_2F_1(b+is, b-is; b+c; -x) x^{b+c-1} (1+x)^{b-c} dx.
\end{aligned}$$

THEOREM 5.3 *The operator  $J_{b,c}^a$  is a unitary operator*

$$W_{b,c}^a \rightarrow L^2 \left( \mathbb{R}_+, \left| \frac{\Gamma(a+is)\Gamma(b+is)\Gamma(c+is)}{\Gamma(2is)} \right|^2 \right).$$

PROOF. The image of the function  $f(z) = z^k$  under the transform  $J_{b,c}^a$  is

$$\frac{S_k(s^2; a, b, c)}{\Gamma(a+b+k)\Gamma(a+c+k)}, \quad (5.3)$$

Thus the required statement follows from (5.2) and the orthogonality relations for the Hahn polynomials.

**5.4. Operator  $d/dz$ .** Denote by  $W^\infty$  the space of functions holomorphic in the disk  $|z| < 1$  and smooth up to the boundary. Let  $a > 1$ . By Lemma 1.1, for  $f \in W^\infty$  the function  $J_{b,c}^a f$  is holomorphic in the strip  $|\operatorname{Im} s| < 1 + \varepsilon$ .

PROPOSITION 5.4. *For each  $f \in W^\infty$  and each  $a > 1$*

$$J_{b,c}^a z \frac{d}{dz} f = \mathcal{L} J_{b,c}^a f,$$

where  $\mathcal{L}$  is the difference operator defined by (4.1).

PROOF. The image of the function  $z^k$  is given by (5.3), and the Hahn polynomials satisfy the difference equations (4.2).

## §6. $\Lambda$ -function and its properties

Motivation of the definition of the  $\Lambda$ -function introduced in this Section is contained in above in 3.7.

**6.1. Definition.** Let  $a, b, c \in \mathbb{R}$ . Let  $x \in \mathbb{R}_+$ . We define the  $\Lambda$ -function by

$$\begin{aligned} \Lambda_{b,c}^a(x) = \frac{1}{\pi\Gamma(b+c)} \int_0^\infty \Gamma(a+is) \frac{\Gamma(b+is)\Gamma(b-is)\Gamma(c+is)\Gamma(c-is)}{\Gamma(2is)\Gamma(-2is)} \times \\ \times {}_2F_1(b+is, b-is; b+c; -x) ds. \end{aligned} \quad (6.1)$$

REMARK. The function  $\Lambda_{b,c}^a(x)$  admits a holomorphic continuation to the domain  $\operatorname{Re} a, \operatorname{Re} b, \operatorname{Re} c > 0$ . It will be more pleasant for us to formulate properties of the  $\Lambda$ -function for real  $a, b, c$ .

I couldn't express the integral (6.1) in terms of standard special functions (see the list in [32]) by a finite number of algebraic operations (except some special values of  $b, c$ , see below 6.6). I think that this is impossible. The integrand in (6.1) is slightly intricate, but the true reason of nontriviality of the integral (6.1) are unusual limits of the integration. Indeed, let us change the limits of integration (for real  $a, b, c$ ) to

$$\int_{-\infty}^{+\infty}. \quad (6.2)$$

This is equivalent to an evaluation of

$$\operatorname{Re} \Lambda_{b,c}^a(x). \quad (6.3)$$

Then our integral becomes a Slater type integral (see [34], [19]) and the standard residue machinery allows to obtain some expansions of  $\operatorname{Re} \Lambda_{b,c}^a(x)$  into series. Nevertheless I couldn't find a nice final expression for  $\operatorname{Re} \Lambda_{b,c}^a(x)$  using this approach.

For some special values of  $b, c$  expression (6.3) can be evaluated explicitly (see 6.7)

Substituting  $x = 0$  to (6.3), we obtain the Barnes integral

$$\operatorname{Re} \Lambda_{b,c}^a(0) = \frac{1}{2\pi\Gamma(b+c)} \int_{-\infty}^{+\infty} \frac{\Gamma(a+is)\Gamma(b+is)\Gamma(b-is)\Gamma(c+is)\Gamma(c-is)}{\Gamma(2is)\Gamma(-2is)} ds.$$

Applying the standard Barnes method, (see [34], [19], [1]) we represent this integral as a sum of two hypergeometric series  ${}_4F_4(1)$

$$\begin{aligned} \operatorname{Re} \Lambda_{b,c}^a(0) &= \sum_0^\infty \frac{(-1)^n \Gamma(a-b-n)\Gamma(2b+n)\Gamma(c-b-n)\Gamma(c+b+n)}{n! \Gamma(2b+2n)\Gamma(-2b-2n)} + \\ &+ \sum_0^\infty \frac{(-1)^n \Gamma(a-c-n)\Gamma(2c+n)\Gamma(b-c-n)\Gamma(b+c+n)}{n! \Gamma(2c+2n)\Gamma(-2c-2n)}. \end{aligned}$$

We observe that  $\Lambda_{b,c}^a(0)$  is some kind of a Barnes integral over a nonclosed contour. Nevertheless the expression  $\Lambda_{b,c}^a(0)$  is 'better' than the 'indefinite Barnes integral'

$$\frac{1}{2\pi\Gamma(b+c)} \int_u^{+\infty} \Gamma(a+is) \frac{\Gamma(b+is)\Gamma(b-is)\Gamma(c+is)\Gamma(c-is)}{\Gamma(2is)\Gamma(-2is)} ds, \quad (6.4)$$

since the point  $s = 0$  is a distinguished point for the second factor of the integrand.

**6.2. Imitation of Ørsted problem.** Consider the space  $W_{b,c}^a$ . The operator  $J_{b,c}^a$  is a unitary operator from  $W_{b,c}^a$  to the space

$$L^2 \left( \mathbb{R}_+, \left| \frac{\Gamma(a+is)\Gamma(b+is)\Gamma(c+is)}{\Gamma(2is)} \right|^2 ds \right). \quad (6.5)$$

The multiplication operator

$$Mg(s) = \Gamma(a+is)g(s) \quad (6.6)$$

is a unitary operator from the space (6.5) to the space

$$L^2 \left( \mathbb{R}_+, \left| \frac{\Gamma(b+is)\Gamma(c+is)}{\Gamma(2is)} \right|^2 ds \right)$$

and the inverse index transform  $(J_{b,c})^{-1}$  is a unitary operator from (6.6) to

$$L^2(\mathbb{R}_+, x^{b+c-1}(1+x)^{b-c}dx).$$

Thus we obtain the unitary map

$$(J_{b,c})^{-1} \circ M \circ J_{b,c}^a : W_{b,c}^a \rightarrow L^2(\mathbb{R}_+, x^{b+c-1}(1+x)^{b-c}dx).$$

This map is an imitation of the operator

$$V_\theta \rightarrow L^2(\mathbb{B}_m(\mathbb{K}))$$

discussed above in 3.7. The function  $\Lambda_{b,c}^a$  is the image of the function  $f(z) = 1$  under this map.

**6.3. Direct corollaries of definition of the  $\Lambda$ -function.** The inversion formula for the index transform implies

$$\frac{1}{\Gamma(b+c)} \int_0^\infty \Lambda_{b,c}^a(x) {}_2F_1(b+is, b-is; b+c; -x) x^{b+c-1}(1+x)^{b-c} dx = \Gamma(a+is).$$

The Plancherel formula for the inverse index transform implies

$$\begin{aligned} \int_0^\infty \Lambda_{b,c}^a(x) \overline{\Lambda_{b,c}^a(x)} x^{b+c-1}(1+x)^{b-c} dx &= \\ &= \frac{1}{\pi} \int_0^\infty \left| \frac{\Gamma(a+is)\Gamma(b+is)\Gamma(c+is)}{\Gamma(2is)} \right|^2 ds = \Gamma(a+b)\Gamma(a+c)\Gamma(b+c). \end{aligned}$$

(for a deduction of the last row see [1], 3.6).

**6.4. Differential-difference equations.** The identity

$${}_2F_1(\alpha, \beta; \gamma; -x) = (1+x)^{\gamma-\alpha-\beta} {}_2F_1(\gamma-\alpha, \gamma-\beta; \gamma; -x)$$

(see [9], vol. 1, 2.1(23)) implies

$$\Lambda_{b,c}^a(x) = (1+x)^{c-b} \Lambda_{c,b}^a(x). \quad (6.7)$$

Differentiating the integral (6.1) in the parameter  $x$  by the formula

$$\frac{d}{dx} {}_2F_1(\alpha, \beta; \gamma; x) = \frac{\alpha\beta}{\gamma} {}_2F_1(\alpha+1, \beta+1; \gamma+1; x),$$

we obtain

$$\begin{aligned} \frac{d}{dx} \Lambda_{b,c}^a(x) &= -\frac{1}{\Gamma(b+c+1)} \int_0^\infty \Gamma(a+is) \left| \frac{\Gamma(b+1+is)\Gamma(c+is)}{\Gamma(2is)} \right|^2 \times \\ &\quad \times {}_2F_1(b+1+is, b+1-is; b+c+1; -x) ds = -\Lambda_{b+1,c}^a(x) \end{aligned}$$

and finally

$$\frac{d}{dx} \Lambda_{b,c}^a(x) = -\Lambda_{b+1,c}^a(x). \quad (6.8)$$

Observe the following corollary from formulas (6.7), (6.8)

$$\Lambda_{b+k,c+l}^a(x) = (-1)^{k+l}(1+x)^{c+l-b-k} \frac{d^l}{dx^l} (1+x)^{b+k-c} \frac{d^k}{dx^k} \Lambda_{b,c}^a(x). \quad (6.9)$$

By the formula (see [1], 2.5.7)

$$\left( x \frac{d}{dx} + \gamma - 1 \right) {}_2F_1(\alpha, \beta; \gamma; x) = (\gamma - 1) {}_2F_1(\alpha, \beta; \gamma - 1; x),$$

we obtain

$$\begin{aligned} & \left( x \frac{d}{dx} + b + c - 1 \right) \Lambda_{b,c}^a(x) = \\ &= -\frac{1}{\Gamma(b+c-1)} \int_0^\infty \Gamma(a+is) \left| \frac{\Gamma(b+is)\Gamma(c-1+is)}{\Gamma(2is)} \right|^2 (c+is)(c-is) \times \\ & \quad \times {}_2F_1(b+is, b-is; b+(c-1); -x) ds. \end{aligned}$$

Representing  $(c+is)(c-is)$  in the form

$$(c+is)(c-is) = (c^2 - a^2) + (2a+1)(a+is) - (a+is)(a+1+is),$$

we obtain

$$\begin{aligned} & \left( x \frac{d}{dx} + b + c - 1 \right) \Lambda_{b,c}^a(x) = \\ &= (c^2 - a^2) \Lambda_{b,c}^a(x) + (2a+1) \Lambda_{b,c}^{a+1}(x) - \Lambda_{b,c}^{a+2}(x) \quad (6.10) \end{aligned}$$

or

$$-x \Lambda_{b+1,c}^a(x) = (c^2 - a^2 - b - c + 1) \Lambda_{b,c}^a(x) + (2a+1) \Lambda_{b,c}^{a+1}(x) - \Lambda_{b,c}^{a+2}(x). \quad (6.11)$$

Certainly our list of difference equations is not complete.

### 6.5. One integral.

PROPOSITION 6.1. For  $n = 0, 1, 2, \dots$

$$\begin{aligned} & 2 \operatorname{Re} \int_0^\infty \Lambda_{b,c}^a(x) \overline{\Lambda_{b,c}^{a+n}(x)} x^{b+c-1} (1+x)^{b-c} dx = \\ &= (2a)_{2n} \Gamma(a+c) \Gamma(b+c) \Gamma(a+b) {}_4F_3 \left[ \begin{matrix} -n/2, -(n+1)/2, a+b, a+c \\ -n+1, 2a, 2a+1 \end{matrix}; 1 \right]. \quad (6.12) \end{aligned}$$

The expression  ${}_4F_3$  in our case is not a series but a finite sum. The right part of (6.12) also can be represented in the form

$$\begin{aligned} & \Gamma(a+b)\Gamma(a+c)\Gamma(b+c)(2a)_{2n} - \\ & \quad - n\Gamma(a+b+1)\Gamma(a+c+1)\Gamma(b+c)(2a+2)_{2n-2} + \\ & + n\Gamma(b+c) \sum_{k=2}^{[(n+1)/2]} \frac{(n-k-1)\dots(n-2k+1)}{k!} \Gamma(a+b+k)\Gamma(a+c+k)(2a+2k)_{2n-2k}. \end{aligned}$$

PROOF. By the Plancherel formula, our integral equals

$$\begin{aligned} & \frac{2}{\pi} \int_0^\infty \operatorname{Re}[\Gamma(a+is)\overline{\Gamma(a+is+n)}] \left| \frac{\Gamma(b+is)\Gamma(c+is)}{\Gamma(2is)} \right|^2 ds = \\ & = \frac{1}{\pi} \int_0^\infty [(a+is)_n + (a-is)_n] \left| \frac{\Gamma(a+is)\Gamma(b+is)\Gamma(c+is)}{\Gamma(2is)} \right|^2 ds. \end{aligned} \quad (6.13)$$

LEMMA 6.2.

$$\begin{aligned} & (a+is)_n + (a-is)_n = \\ & = (2a)_{2n} - n(2a+2)_{2n-2}(a+is)(a-is) + \\ & \quad + n \sum_{k=2}^{[(n+1)/2]} \frac{(-1)^k(-n+k+1)_{k-1}}{k!} (2a+2k)_{2n-2k} (a+is)_k (a-is)_k = \\ & = (2a)_{2n} \sum_{j=0}^{[(n+1)/2]} \frac{(-n/2)_j(-(n+1)/2)_j (a-is)_j (a+is)_j}{j! (-n+1)_j (2a)_{2j}}. \end{aligned}$$

By Lemma 6.2, we write integral (6.13) in the form

$$\begin{aligned} & (2a)_{2n} \sum_{k=0}^{[(n+1)/2]} \frac{(-n/2)_j(-(n+1)/2)_j}{j! (-n+1)_j (2a)_{2j}} \times \\ & \quad \times \int_0^\infty \left| \frac{\Gamma(a+j+is)\Gamma(b+is)\Gamma(c+is)}{\Gamma(2is)} \right|^2 ds = \\ & = (2a)_{2n} \sum_{k=0}^{[(n+1)/2]} \frac{(-n/2)_j(-(n+1)/2)_j}{j! (-n+1)_j (2a)_{2j}} \Gamma(b+c)\Gamma(a+j+b)\Gamma(a+j+c). \end{aligned}$$

This implies the required result.

**6.6. Relations with the  $\lambda$ -function.** The function  $\lambda(z, a)$  is given by

$$\lambda(z, a) := \int_0^a z^{-t} \Gamma(t+1) dt.$$

It will be more convenient for us to change notations and to consider the function

$$\lambda^*(z, b) = \int_b^{i\infty} z^{-t} \Gamma(t+1) dt$$

(it is the same indefinite integral with another point of the origin). The function  $\lambda$  is an element of a relatively exotic family of special functions (sometimes they are called the *Volterra type functions*). This family includes also the functions

$$\begin{aligned}\mu(z, p) &= \int_0^\infty \frac{t^p z^t dt}{\Gamma(t+1)}; & \mu(z, p, \rho) &= \int_0^\infty \frac{t^p z^{t+\rho} dt}{\Gamma(\rho+t+1)}; \\ \nu(z) &= \int_0^\infty \frac{z^t dt}{\Gamma(t+1)}; & \nu(z, \rho) &= \int_0^\infty \frac{z^{t+\rho} dt}{\Gamma(\rho+t+1)}.\end{aligned}$$

Observe some superficial resemblance of these functions with (6.1). Indeed, these integrals are similar to the Barnes integrals, but we have 'incorrect' limits of the integration. The functions  $\mu(z, p)$ ,  $\mu(z, p, \rho)$  were appeared in the Volterra work ([37]) on the fractional derivatives with logarithmic terms (see also [33]). The functions  $\mu$ ,  $\nu$  were intensively discussed in French and Belgian mathematical journals in the first half of 1940ies (Humbert, Poli, Colombo, Parodi and others, for instance, see, [14], [4]). The first mention on the  $\lambda$ -function that I find is contained in the tables of McLachlan, Humbert, Poli [20].

A theory of the functions  $\nu$ ,  $\mu$  is contained in 'Higher transcendent functions' [9], 18.3 (the book contains also a large bibliography on this subject), The integrals with the functions  $\nu$ ,  $\mu$ ,  $\lambda$  can be find in the tables of Prudnikov, Brychkov, Marichev, see corresponding subsections in volumes 3–5.

Let us show that the functions

$$\Lambda_{1/2+k,l}^a(x), \quad \Lambda_{k,1/2+l}^a(x)$$

can be expressed in terms of  $\lambda^*(z, b)$  (in particular, this is valid for the  $\Lambda$ -functions related with groups of the subseries  $O(1, 2n+1)$ ). For instance, consider the function

$$\begin{aligned}\Lambda_{1/2,1}^a(x) &:= \\ &= \frac{1}{\pi \Gamma(3/2)} \int_0^\infty \Gamma(a+is) \left| \frac{\Gamma(1/2+is)\Gamma(1+is)}{\Gamma(2is)} \right|^2 \frac{1}{2s\sqrt{x}} \sin(2s \operatorname{arcsh} \sqrt{x}) ds = \\ &= -\frac{4}{\sqrt{\pi x}} \int_0^\infty \Gamma(a+is) is [e^{2is \operatorname{arcsh} \sqrt{x}} - e^{-2is \operatorname{arcsh} \sqrt{x}}] ds = \\ &= -\frac{4}{\sqrt{\pi x}} \int_0^\infty (\Gamma(a+1+is) - a\Gamma(a+is)) \times \\ &\quad \times [(\sqrt{x^2+1} + x)^{2is} - (\sqrt{x^2+1} - x)^{2is}] ds.\end{aligned}$$

We obtained a sum of 4 integrals that can be expressed by the function  $\lambda^*$ .

Using formula (6.9), we obtain an expression of  $\Lambda_{1/2+k,1+l}^a(x)$  in terms of the derivatives of  $\lambda^*(x)$ . The case  $\Lambda_{k,1/2+l}^a$  is similar.

Observe that the derivatives of the function  $\lambda^*$  can be expressed algebraically by the same function  $\lambda^*(x)$

$$\begin{aligned}\frac{d}{dz}\lambda^*(z, b) &= - \int_b^{+i\infty} \frac{t}{z} z^{-t} \Gamma(t+1) dt = \\ &= -\frac{1}{z} \int_b^{+i\infty} (\Gamma(t+2) - \Gamma(t+1)) z^{-t} dt = -\lambda^*(z, b+1) + \frac{1}{z} \lambda^*(z, b).\end{aligned}$$

**6.7. The cases of explicit evaluation for  $\operatorname{Re} \Lambda_{b,c}^a$ .** Let  $k, l = 0, 1, 2, \dots$ . Let us show that the following functions can be evaluated explicitly

$$\operatorname{Re} \Lambda_{1/2+k,l}^a(x), \quad \operatorname{Re} \Lambda_{k,1/2+l}^a(x).$$

Let us calculate

$$\begin{aligned}\operatorname{Re} \Lambda_{1/2,1}^a &:= \operatorname{const} \int_{-\infty}^{+\infty} \Gamma(a+is) s^2 \frac{1}{2s\sqrt{x}} \sin(2s \operatorname{arcsh} \sqrt{x}) ds = \\ &= \operatorname{const} \int_{-\infty}^{+\infty} (\Gamma(a+1+is) - a\Gamma(a+is)) \times \\ &\quad \times [(\sqrt{x^2+1}+x)^{2is} - (\sqrt{x^2+1}-x)^{2is}] ds.\end{aligned}$$

Using the formula (see, for instance, [1], (2.4.1))

$$\frac{1}{2\pi i} \int_{-i\infty}^{+i\infty} \Gamma(u+t) z^t dt = z^u e^{-z},$$

we obtain

$$\begin{aligned}\operatorname{const} \left[ (x + \sqrt{x^2+1})^{2a} ((x + \sqrt{x^2+1})^2 - a) \exp\{-(x + \sqrt{x^2+1})^2\} - \right. \\ \left. - (\sqrt{x^2+1} - x)^{2a} ((\sqrt{x^2+1} - x)^2 - a) \exp\{-(\sqrt{x^2+1} - x)^2\} \right].\end{aligned}$$

To obtain an explicit expression for  $\operatorname{Re} \Lambda_{1/2+k,l}^a(x)$ , we can apply formula (6.9).

REMARK. Slight inconvenience of the formulas of this and the previous subsection disappears after the substitution

$$x = \operatorname{sh}^2 t.$$

**6.8. Comments.** At the end of §3, I emphasized that there is some arbitrariness in the choice of the  $\Lambda$ -function. Some of properties of our list survive for arbitrary choice of the function  $\psi$  ( (3.10)), i.e. for general integrals having the form

$$\begin{aligned}\frac{1}{\pi \Gamma(b+c)} \int_0^\infty \Gamma(a+is) e^{iu(s)} \frac{\Gamma(b+is)\Gamma(b-is)\Gamma(c+is)\Gamma(c-is)}{\Gamma(2is)\Gamma(-2is)} \times \\ \times {}_2F_1(b+is, b-is; b+c; -x) ds,\end{aligned}$$

where  $\operatorname{Im} u(s) = 0$ . This is valid for integrals from Subsection 6.3, for differential-difference equations (6.7)-(6.9), and for Theorem 7.1, Proposition 7.2, Theorem 8.1.

It seems important, that the  $\Lambda$ -function has a collection of properties that are not corollaries of the motivation given in 3.7, these properties are valid only for  $\psi = 1$  (or  $u = 0$ ). This is the difference equation (6.10) and all material of 6.5-6.7.

It should be noted that the differential-difference equations obtained in 6.4 and the expressions by the  $\lambda$ -function are valid also for the indefinite Slater integral (6.4). All other properties of the  $\Lambda$ -function obtained above don't survive in this generality.

## §7. Bireflected bases and generalized translate hypergroup

**7.1. Bireflected bases.** In 5.3 we defined the unitary operator

$$Q = (J_{b,c})^{-1} \circ M \circ J_{b,c}^a : W_{b,c}^a \rightarrow L^2(\mathbb{R}_+, x^{b+c-1}(1+x)^{b-c}dx).$$

By

$$\Xi_n(x) = \Xi_n(x; a, b, c)$$

we denote the image of the function  $z^n$  under the transform  $Q$ .

**THEOREM 7.1.** a) *The system of the functions  $\Xi_n$  forms an orthogonal basis in  $L^2(\mathbb{R}_+, x^{b+c-1}(1+x)^{b-c}dx)$ .*

b) *The functions  $\Xi_n$  can be expressed in terms of the  $\Lambda$ -function by*

$$\begin{aligned} \Xi_n(x; a, b, c) &= \\ &= \frac{(b+c)_n}{\Gamma(a+c)\Gamma(a+b+n)} \sum_{j=0}^n \frac{(-1)^j (-n)_j}{j! (a+c)_j (b+c)_j} x^{-b-c+1} \frac{d^j}{dx^j} x^{b+c+j-1} \Lambda_{b,c+j}^a. \end{aligned}$$

**PROOF.** The statement a) is obvious and we must prove only b). Recall that the image of the function  $z^n$  under the operator  $J_{b,c}^a$  is a Hahn polynomial. Thus it is sufficient to evaluate the integral

$$\begin{aligned} &\frac{1}{\Gamma(a+b+n)\Gamma(a+c+n)\Gamma(b+c)} \int_0^\infty \Gamma(a+is) \left| \frac{\Gamma(b+is)\Gamma(c+is)}{\Gamma(2is)} \right|^2 \times \\ &\quad \times {}_2F_1(b+is, b-is; b+c; -x) S_n(x^2; a, b, c) ds. \end{aligned}$$

By the symmetry of the Hahn polynomials in  $a, b, c$ , we represent the integral in the form

$$\begin{aligned} &\frac{1}{\Gamma(a+b+n)\Gamma(a+c+n)\Gamma(b+c)} \int_0^\infty \Gamma(a+is) \left| \frac{\Gamma(b+is)\Gamma(c+is)}{\Gamma(2is)} \right|^2 \times \\ &\quad \times {}_2F_1(b+is, b-is; b+c; -x) (a+c)_n (b+c)_n {}_3F_2 \left( \begin{matrix} -n, c+is, c-is \\ a+c, b+c \end{matrix}; 1 \right) ds. \end{aligned}$$

Expanding  ${}_3F_2$  into the sum, we obtain

$$\begin{aligned} & \frac{(b+c)_n}{\Gamma(a+b+n)\Gamma(a+c)\Gamma(b+c)} \sum_{j=0}^n \frac{(-n)_j}{j!(a+c)_j(b+c)_j} \times \\ & \times \int_0^\infty \Gamma(a+is) \left| \frac{\Gamma(b+is)\Gamma(c+j+is)}{\Gamma(2is)} \right|^2 {}_2F_1(b+is, b-is; b+c; -x) ds. \end{aligned}$$

Applying the formula ([9], vol.1, 2.8(22))

$$(\mu)_k y^{\mu-1} {}_2F_1(\alpha, \beta; \mu; y) = \frac{d^k}{dy^k} y^{\mu+k-1} {}_2F_1(\alpha, \beta; \mu+k; y),$$

we obtain

$$\begin{aligned} & \frac{(b+c)_n}{\Gamma(a+b)\Gamma(a+c)} \sum_{j=0}^n \frac{(-n)_j}{j!(a+c)_j(b+c)_j} \frac{1}{\Gamma(b+c+j)} (-1)^j x^{1-b-c} \frac{d^j}{dx^j} x^{b+c+j-1} \times \\ & \times \int_0^\infty \Gamma(a+is) \left| \frac{\Gamma(b+is)\Gamma(c+j+is)}{\Gamma(2is)} \right|^2 {}_2F_1(b+is, b-is; b+(c+j); -x) ds. \end{aligned}$$

This gives the required expression.

**7.2. Generalized translate operator.** The generalized translate hypergroup is an essential element of the theory of the index hypergeometric transform. (see [10], [17]). We only will slightly touch on this subject.

Consider the set  $J \subset \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}_+$  consisting of all points  $(x, y, z)$  such that three numbers  $\operatorname{arcsh} \sqrt{x}, \operatorname{arcsh} \sqrt{y}, \operatorname{arcsh} \sqrt{z}$  satisfy the triangle inequality. Consider the function  $K(x, y, z)$  on  $\mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}_+$  that is identical zero outside the set  $J$  and is equal to

$$\begin{aligned} K(x, y, z) = & \frac{2^{-4b}\Gamma(b+c)}{\Gamma(b+c-1/2)} \frac{((1+x)(1+y)(1+z))^{c-1}}{(xyz)^{b+c-1}} \times \\ & \times (1-B^2)^{b+c-1/2} {}_2F_1(2b-1, 2c-1; b+c-1/2; (1-B)/2) \end{aligned}$$

on the set  $J$ , where  $B$  denotes the expression

$$B = \frac{x+y+z+2}{2(1+x)(1+y)(1+z)}.$$

The *generalized translate operator*  $T_z : L^2(\mathbb{R}_+, x^{b+c-1}(1+x)^{b-c})$  is given by

$$T_z f(y) = \int_0^\infty K(x, y, z) f(x) x^{b+c-1}(1+x)^{b-c} dx.$$

It is known that

$$J_{b,c} T_z (J_{b,c})^{-1} g(s) = {}_2F_1(b+is, b-is; b+c; -z) f(s). \quad (7.1)$$

PROPOSITION 7.1.

$$\begin{aligned} \int_0^\infty \int_0^\infty \Lambda_{b,c}^a(x) \overline{\Lambda_{b,c}^a(y)} K(x, y, z) x^{b+c-1} (1+x)^{b-c} y^{b+c-1} (1+y)^{b-c} dx dy = \\ = \Gamma(a+b)\Gamma(a+c)\Gamma(b+c)(1+z)^{-a-b}. \end{aligned}$$

PROOF. The expression in the left side is the scalar product  $\Lambda_{b,c}^a T_z \Lambda_{b,c}^a$  in  $L^2(\mathbb{R}_+, x^{b+c-1}(1+x)^{b-c})$ . By the Plancherel formula (7.1), this equals

$$\int_0^\infty \left| \frac{\Gamma(b+is)\Gamma(c+is)}{\Gamma(2is)} \right|^2 \Gamma(a+is) \overline{\Gamma(a+is)} {}_2F_1(b+is, b-is; b+c; -z) ds.$$

The last expression is the inverse index transform of  $|\Gamma(a+is)|^2$ . But it was evaluated in Lemma 4.1.

**7.3. Generalized translate operator and bireflected bases.** It seems that the generalized translate operator has a simple explicit matrix in the bireflected basis.

Author can evaluate only the first row.

PROPOSITION 7.3.

$$T_z \Lambda_{b,c}^a = T_z \Xi_0(x; a, b, c) = (1+z)^{a-b} \sum_{n=0}^{\infty} \frac{1}{n!(b+c)_n} \left( \frac{z}{z+1} \right)^n \Xi_n(x; a, b, c).$$

PROOF. The index transform of  $T_z \Lambda_{b,c}^a$  is

$$\begin{aligned} \Gamma(a+is) {}_2F_1(b+is, b-is; b+c; -z) &= \\ &= \Gamma(a+is)(1+z)^{-b-is} {}_2F_1(b+is, c+is; b+c; \frac{z}{z+1}) = \\ &= \Gamma(a+is)(1+z)^{a-b} \left( 1 - \frac{z}{z+1} \right)^{a+is} {}_2F_1(b+is, c+is; b+c; \frac{z}{z+1}). \end{aligned}$$

Using the following generating function for the continuous dual Hahn polynomials (see [1], p.349), we obtain

$$\Gamma(a+is)(1+z)^{a-b} \sum_{k=0}^{\infty} \frac{S_k(x^2; a, b, c)}{k! (b+c)_k} \left( \frac{z}{z+1} \right)^k.$$

This is equivalent to the required statement.

## §8. An application of $\Lambda$ -function and of bireflected bases. Symmetric spaces $U(p, q)/U(p) \times U(q)$

**8.1. The space  $U(p, q)/U(p) \times U(q)$ .** Assume  $p \leq q$ . Denote by  $U(p, q)$  the pseudounitary group of the order  $(p, q)$ , i.e. the group of all matrices  $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ , satisfying the condition

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix}^* = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Denote by  $B_{p,q}$  the space of all matrices  $z$  over  $\mathbb{C}$  having the size  $p \times q$  with the norm satisfying  $\|z\| < 1$ . The group  $U(p, q)$  acts on  $B_{p,q}$  by the linear fractional transforms

$$z \mapsto z^{[g]} := (a + zc)^{-1}(b + zd).$$

The stabilizer  $K$  of the point 0 consists of matrices having the form

$$\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}; \quad a \in U(p), d \in U(q).$$

The space  $B_{p,q}$  is a homogeneous symmetric space,

$$B_{p,q} = U(p, q)/U(p) \times U(q).$$

The  $U(p, q)$ -invariant measure on  $B_{p,q}$  has the density  $\det(1 - zz^*)^{-p-q}$  with respect to the Lebesgue measure. The group  $U(p, q)$  acts in the space  $L^2$  with respect to the invariant measure by the substitutions

$$T(g)f(z) = f(z^{[g]}). \quad (8.1)$$

**8.2. Berezin spaces  $H_\theta^\circ(B_{p,q})$ .** The Berezin space  $V_\theta^\circ(B_{p,q})$  is the space of functions on  $B_{p,q}$  defined by the kernel

$$K(z, u) = \frac{\det(1 - zz^*)^{\theta/2} \det(1 - uu^*)^{\theta/2}}{|\det(1 - zu^*)|^\theta}.$$

The group  $U(p, q)$  acts in  $V_\theta^\circ(B_{p,q})$  by formula (8.1). For sufficiently large  $\theta$  the representation of  $U(p, q)$  in  $V_\theta^\circ(B_{p,q})$  is equivalent to the representation in  $L^2(B_{p,q})$  (see [2], see also [29], [26]). The *Ørsted problem* is to construct explicitly a unitary intertwining operator between these representations.

**8.3.  $\Lambda$ -function of the space  $U(p, q)/U(p) \times U(q)$ .** Denote by  $\lambda_1(z), \dots, \lambda_p(z)$  the singular values of the matrix  $z$ . Define the new variables by

$$x_j(z) = \lambda_j^2(z)/(1 - \lambda_j^2(z)).$$

We define the  $\Lambda$ -function of the space  $U(p, q)/U(p) \times U(q)$  by

$$\Lambda^\theta(z) = \det \begin{pmatrix} \Xi_0(x_1) & \dots & \Xi_0(x_p) \\ \Xi_1(x_1) & \dots & \Xi_1(x_p) \\ \vdots & \ddots & \vdots \\ \Xi_{p-1}(x_1) & \dots & \Xi_{p-1}(x_p) \end{pmatrix}, \quad (8.2)$$

where

$$\Xi_j(x) = \Xi_j(x; \theta - (q + p - 1)/2; (q - p + 1)/2; (q - p + 1)/2)$$

are the first  $p$  elements of the bireflected basis.

THEOREM 8.1. For  $g \in U(p, q)$  by  $u$  we denote the image of the point  $0 \in B_{p,q}$  under  $g$ . Then

$$\int_{B_{p,q}} \Lambda^\theta(z) \overline{\Lambda^\theta(z^{[g]})} \det(1 - zz^*)^{-p-q} dz = \det(1 - uu^*)^{\theta/2}.$$

COROLLARY 8.2. Define the function  $L(z, u)$  on  $B_{p,q} \times B_{p,q}$  by the formula

$$L(z, u) = \Lambda(z^{[g_u]}),$$

where  $g_u \in U(p, q)$  is any element that transforms the point  $0 \in B_{p,q}$  to  $u$ . Then the operator

$$Mf(z) = \int_{B_{p,q}} L(z, u) f(u) \det(1 - uu^*)^{-p-q} du$$

is a unitary  $U(p, q)$ -intertwinning operator

$$L^2(B_{p,q}) \rightarrow V_\theta^o(B_{p,q}).$$

REMARK. The definition of the kernel  $L(z, u)$  can be reformulated in the following form. It is defined by the determinant (8.2), where the numbers  $\lambda_j$  are the singular values of the matrix

$$(1 - zz^*)^{-1/2} (1 - zu^*)(1 - uu^*)^{-1/2}.$$

REMARK. The similar statement is valid for the groups of series  $O(n, \mathbb{C})$ ,  $Sp(2n, \mathbb{C})$ .

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INSTITUTE OF THEORETICAL AND EXPERIMENTAL PHYSICS  
 neretin@main.mccme.rssi.ru